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VAN DER CORPUT'S DIFFERENCE THEOREM

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ABSTRACT

We obtain a sufficient condition for a subset H of positive integers to satisfy that the equidistribution (mod 1) of the sequences $(u_{n+h} - u_n;\ n = 1, 2, \cdots)$ for all $h \in H$ implies the equidistribution of (u_n). Our condition is satisfied, for example, for the following sets: (1) $H = \{n - m;\ n \in I,\ m \in I,\ n > m\}$, where I is any infinite subset of integers; (2) $H = \{|\psi(n)|;\ \psi(n) \neq 0,\ n \in Z\}$, where ψ is a nonconstant polynomial with integral coefficients having at least one integral zero (mod q) for all $q = 2, 3, \cdots$; (3) $H = \{p + 1;\ p$ is a prime$\}$ and $H = \{p - 1;\ p$ is a prime$\}$.

§1. Introduction

We wish to illustrate in this article the relationship between the properties of positive definite sequences and that of the distribution (mod 1) of real sequences. The idea goes back to Bass [1], Bertrandias [2] and Cigler [3]. (Another kind of link was discovered by Elliott [6] and Niederreiter and Philipp [11] some years later.) Our concern is mainly that of the first mentioned authors who have used a harmonic analytic proof of the celebrated van der Corput theorem according to which *the equidistribution* (mod 1) *of the sequences* $(u_{n+h} - u_n)$ *for* $h = 1, 2, 3, \cdots$ *implies the equidistribution of* (u_n). Several extensions of this result are known. For example, if $(u_{n+h} - u_n)$ is equidistributed (mod 1) for all $h \neq 0$ of a set of density 1, then again (u_n) is equidistributed (mod 1) together with the subsequences (u_{an+b}), $a \in \mathbf{N} = \{1, 2, 3, \cdots\}$. The result also holds if $h \neq 0$ runs through the set of multiples of a positive integer (a result of Delange, see [2]).

We shall exhibit here a large class of zero density sequences with the van der Corput property: if $(u_{n+h} - u_n)$ is equidistributed (mod 1) for all $h \in H$, then (u_{an+b}) is equidistributed (mod 1) for all integers $a \geqq 1$ and b. In particular, if I is any infinite set of integers

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$$H = \{n - m \mid n \in I,\ m \in I,\ n > m\} \quad \text{and} \quad H = \{n^2 \mid n \in \mathbf{N}\}$$

share the van der Corput property. Another example is the set of "primes minus one" (or plus one). The three last examples are in close relationship with deep results due to Sárközy [16], [17], [18].

§2. A property of positive definite sequences

Let $\gamma : \mathbf{Z} \to \mathbf{C}$ be a nonzero positive definite sequence, with $|\gamma(0)| \leqq 1$. In other words, a sequence $\gamma = (\gamma(n))$ for which there exists a positive measure Λ with $\Lambda(T) \leqq 1$ defined on the torus $T = \mathbf{R}/\mathbf{Z}$ satisfying

$$\gamma(n) = \int_T \exp 2i\pi nx\, \Lambda(dx), \qquad n \in \mathbf{Z}.$$

We shall write equivalently

$$\Lambda(dx) \sim \sum_{n \in \mathbf{Z}} \gamma(n) \exp(-2i\pi nx).$$

Let $H \subset \mathbf{N}$ and let f be an exponential polynomial, the spectrum of which is contained in H:

$$f(x) = \sum_{h \in H} \hat{f}(h) \exp 2i\pi hx,$$

where

$$\hat{f}(n) = \int_T f(x) \exp(-2i\pi nx) dx, \qquad n \in \mathbf{Z}$$

is nonzero for only finitely many n's all of which lie in H. It is then clear that

$$\int_T f(x)\Lambda(dx) = \sum_{h \in H} \hat{f}(h)\gamma(h), \tag{1}$$

a finite sum.

Let $P(H)$ be the set of exponential polynomials f with spectrum in H, such that $f(0) = 1$ and $|f(x)| \leqq 1$ for every $x \in T$. Denote by $P^*(H)$ the minimum family of functions on T containing $P(H)$, which is closed under the pointwise convergence of functions.

Let $\delta \geqq 0$; $P^*_\delta(H)$ represents the possibly empty set of f's in $P^*(H)$ such that

$$\operatorname{Re}\{f(x)\} \geqq -\delta \qquad \text{for all } x \in T.$$

Suppose $P^*_\delta(H) \neq \varnothing$ and let $f \in P^*_\delta(H)$. Then obviously

$$\int_T \operatorname{Re}\{f(x)\}\Lambda(dx) \geqq \Lambda(\{0\}) - \delta.$$

Assume that γ vanishes on H. Then by (1)

$$\int_T g(x)\Lambda(dx) = 0 \qquad \text{for any } g \in P(H).$$

Therefore

$$\int_T g(x)\Lambda(dx) = 0 \qquad \text{for any } g \in P^*(H).$$

Applying this for f, we then have $\Lambda(\{0\}) \leqq \delta$.

We have thus proved the following result.

THEOREM 1. *Let γ be a nonzero positive definite sequence with spectral measure Λ such that $|\gamma(0)| \leqq 1$. Let $H \subset \mathbf{N}$ be a set of integers such that both conditions hold:*

(i) *for some $\delta \geqq 0$, $P_\delta^*(H) \neq \varnothing$,*

(ii) $\forall h \in H$, $\gamma(h) = 0$.

Then

$$\Lambda(\{0\}) \leqq \delta.$$

§3. An application to van der Corput's difference theorem

Before describing sets H which satisfy conditions of the theorem we shall state and establish a result related to the theory of equidistribution (mod 1).

COROLLARY. *Let (u_n) be an infinite sequence of real numbers reduced* (mod 1). *Let $H \subset \mathbf{N}$ be a set of integers for which the sequence $(u_{n+h} - u_n)$ $(h \in H)$ is equidistributed* (mod 1). *If $P_\delta^*(H) \neq \varnothing$ for all $\delta > 0$. then (u_n) is equidistributed* (mod 1) *and so are the sequences (u_{an+b}), $a \in \mathbf{N}$, $b \in \mathbf{Z}$.*

PROOF. Let $\alpha : \mathbf{N} \to \mathbf{C}$ be a sequence such that $\alpha(n) = o(\sqrt{n})$ and

$$\|\alpha\|^2 = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(n)|^2 \leqq 1.$$

There exists a sequence $S \subset \mathbf{N}$ for which the following limits exist:

$$\lim_{N\in S} \frac{1}{N} \left| \sum_{n=1}^{N} \alpha(n) \right| = \limsup_{N\to\infty} \frac{1}{N} \left| \sum_{n=1}^{N} \alpha(n) \right|$$

and

$$\lim_{N\in S}\frac{1}{N}\sum_{n=1}^{N}\overline{\alpha(n)}\alpha(n+q)=\gamma(q),\qquad \forall q\in\mathbf{Z}.$$

It is well known that γ is positive definite (see [1], [2], [3] or [9]). Let Λ be the associated spectral measure. By corollary 2 in [4], or [2]

$$\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}\alpha(n)\right|\leqq[\Lambda(\{0\})]^{1/2}.$$

Let $H\subset\mathbf{N}$ be the set of zeros of γ. If for all $\delta>0$, $P_\delta^*(H)\neq\varnothing$, then, according to the theorem $\Lambda(\{0\})=0$:

$$\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}\alpha(n)\right|=0.$$

Choose $\alpha(n)=\exp 2i\pi k u_n$, $k\in\mathbf{Z}^*$. Weyl's criterion implies the equidistribution (mod 1) of (u_n).

Observe now that the equidistribution (mod 1) of $(u_{n+h}-u_n)$ is the same as that of $(u_{n+h}-u_n+hx)$, hence the hypothesis also implies the equidistribution (mod 1) of (u_n+nx) for all $x\in T$. Standard techniques developed in [5], [10] and [14] then show that (u_{an+b}) is equidistributed (mod 1) for all $a\in\mathbf{N}$, $b\in\mathbf{Z}$.

Remark. Actually the conclusion can be widened. Let $\mathcal{R}$ be the family of sequences (δ_n) such that $(v_n+\delta_n)$ is equidistributed (mod 1) if and only if (v_n) is (this class is discussed by Rauzy in [12], [13] and by Rindler in [15]). Let $\Sigma\mathcal{R}$ be the set of sequences (φ_n) such that $(\varphi_{n+1}-\varphi_n)\in\mathcal{R}$: for example $\varphi_n=xn$ or $\varphi_n=xn\log n$. Then under the conditions of the corollary, not only is (u_n) equidistributed (mod 1) but also $(u_n+\varphi_n)$ for all $(\varphi_n)\in\Sigma\mathcal{R}$. Again, using results from [5] or [10], if (m_n) is an increasing sequence of integers such that $(m_n)=O(n)$ and such that its characteristic function is almost periodic in the sense of Besicovitch, it can easily be shown that $(u_{m_n}+\varphi_{m_n})$ is equidistributed (mod 1).

§4. Three examples

Example 1. $H=\mathbf{N}$. Define

$$f_N(x)=\frac{1}{N}\sum_{n=1}^{N}\exp 2i\pi xn$$

$$=\frac{\sin\pi xN}{N\sin\pi x}\exp(i\pi(N+1)x)\qquad\text{if } x\neq 0.$$

Hence

$$f(x)=\lim_{N\to\infty} f_N(x)=\begin{cases}0 & \text{if } x\neq 0,\\ 1 & \text{if } x=0.\end{cases}$$

Thus $f\in P_0^*(\mathbf{N})$ so $P_0^*(\mathbf{N})\neq\emptyset$. The corollary applied to $\mathbf{N}$ is then van der Corput's classic theorem.

EXAMPLE 2. Let I be any infinite set of integers. Then $H=(I-I)^*=\{p-q \mid p\in I,\ q\in I,\ p>q\}$ verifies the property $P_\delta^*(H)\neq\emptyset$ for any $\delta>0$. Indeed, suppose J is a finite subset of I and let

$$|J|=\operatorname{card} J\geqq 1+1/\delta.$$

Define the function f_J:

$$f_J(x)=\frac{2}{|J|^2-|J|}\sum_{\substack{p\in J\\ q\in J\\ p>q}}\exp 2i\pi x(p-q).$$

Then

$$\operatorname{Re}\{f_J(x)\}=\frac{1}{|J|^2-|J|}\left|\sum_{q\in J}\exp 2i\pi qx\right|^2-\frac{1}{|J|-1}$$

$$\geqq -\frac{1}{|J|-1}\geqq-\delta.$$

Thus $f_J\in P_\delta^*(H)$. Hence H is a van der Corput set: the equidistribution (mod 1) of $(u_{n+p}-u_{n+q})$ for all $p\in I$, $q\in I$, $p\neq q$ implies the equidistribution (mod 1) of (u_{an+b}) for all integers $a\geqq 1$ and b.

EXAMPLE 3 (suggested by Sárközy's results [16], [17], [18] and Furstenberg's [8] theorem 1.2, p. 208). Let $H\subset\mathbf{N}$. For every $q\in\mathbf{N}$, define

$$H_q=\{h\in H \mid h\equiv 0 \pmod{q!}\}.$$

If for infinitely many q, the sequence xH_q is equidistributed (mod 1) for all irrational x (H_q is hence necessarily infinite), then H is a van der Corput set.

Before establishing our claim, let us give two examples of such an H. First let $\psi\in\mathbf{Z}[X]$ be a nonconstant polynomial which has at least one integral zero (mod q) for all $q\in\mathbf{N}$. Then

$$H(\psi)=\{|\psi(n)| \mid \psi(n)\neq 0,\ n\in\mathbf{Z}\}$$

satisfies the above condition. Using our Theorem 2 (below), it is easy to check that if ψ has no integral zero (mod q) for some q then $H(\psi)$ is not a van der Corput set.

Another example is the set of primes plus 1, or the set of primes minus 1 (this follows from Vinogradov's theorem on the equidistribution (mod 1) of the sequence (xp) where x is irrational and where p runs through the set of primes: see [7], p. 347). Again, Theorem 2 implies that the primes plus k is not a van der Corput set if k is different from ± 1.

We now prove the claim. Let

$$f_{N,q}(x) = \frac{1}{A(N,q)} \sum_{\substack{n \leqq N \\ h_n \equiv 0 \, (\mathrm{mod}\, q!)}} \exp 2i\pi x h_n,$$

where $A(N,q)$ denotes the number of terms in the sum. For any irrational x,

$$\lim_{N \to \infty} f_{N,q}(x) = 0$$

by Weyl's criterion. Since $\mathbf{Q}$ is countable, we can select a subsequence $\{N'\}$ of $\mathbf{N}$ such that $f_{N',q}(x)$ converges for every $x \in \mathbf{T}$ as N' goes to infinity. Let $g_q(x)$ be the limit. Quite obviously,

$$j(x) = \lim_{q \to \infty} g_q(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Thus $j \in P_0^*(H)$ and consequently H has the van der Corput property.

§5. Negative results

We shall now give a condition which enables us to construct sets which are not van der Corput sets. For any sequence $S \subset \mathbf{N}$ we define the upper density

$$\bar{d}(S) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_S(n)$$

where χ_S is the characteristic function of S.

THEOREM 2. *If there exists a set A with nonzero upper density such that $(A - A) \cap H = \varnothing$, then H cannot be a van der Corput set.*

PROOF. Let $u = (u_n)$ be an infinite sequence of real numbers (mod 1) which vanishes on A:

$$n \in A \Rightarrow u_n = 0.$$

The sequence u cannot be equidistributed (mod 1). Let $k \in \mathbf{N}$. Then

$$u_{n+k}-u_n=\begin{cases} 0 & \text{if } n\in A \text{ and } n+k\in A,\\ -u_n & \text{if } n\notin A \text{ and } n+k\in A,\\ u_{n+k} & \text{if } n\in A \text{ and } n+k\notin A,\\ u_{n+k}-u_n & \text{if } n\notin A \text{ and } n+k\notin A.\end{cases}$$

If $k\in H$, the two conditions $n\in A$ and $n+k\in A$ cannot hold. It can then easily be seen that for almost all $u\in\mathbf{T}^{\mathbf{N}}$, the sequences $(u_{n+k}-u_n)$ are equidistributed (mod 1) for all $k\in H$.

Using results of C. Stewart and R. Tijdeman [19] one can then give examples of sequences H which do not have the van der Corput property. Such are

(1) Finite unions of arithmetic progressions which do not contain all the multiples of some positive integer.

(2) Lacunary sequences $H=\{h_1<h_2<\cdots\}$. More precisely sequences such that there exists an $i\in\mathbf{N}$ for which

$$\liminf_{n\to\infty} h_{n+i}/h_n>1.$$

§6. Final remark

It seems interesting to study the family of sequences $K\subset\mathbf{N}$ which have the following property. There are no positive discrete bounded measures Λ such that

$$\lim_{\substack{k\to\infty\\ k\in K}}\int_T \exp 2i\pi kx\,\Lambda(dx)=0.$$

These sets are obviously related to the van der Corput sets. (Katznelson denotes this family by FC^+: the above condition *forces* the *continuity* of *positive* measures.)

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